Yau College Math Competition 2024
Final Probability and Statistics
Individual Exam Problems (June 8-9, 2024)

Choose **at least** 3 from the following 4 problems.

Problem 1. Let $\{X_i\}_{i\geq 0}$ be iid with density function f and distribution function F. Define $N = \min\{n \geq 1 : X_n > X_0\}$.

(1) Find the distribution function of X_N.

(2) If $\mathbb{E}|X_0| < \infty$, is it always true that $\mathbb{E}|X_N| < \infty$? If yes, prove it; if not, give a counterexample.

Problem 2. A fair coin is tossed repeatedly and independently, and the outcome is denoted as $X_1X_2\cdots$ with $X_i = H$ (head) or T (tail).

(1) Describe an idea about how to find the expected number of tosses required until a particular pattern of heads/tails appears.

(2) Evaluate the expected number of tosses to get the special pattern $HTHH$, i.e., evaluate $\mathbb{E}(N)$, where $N = \min\{n \geq 4 : X_{n-3}X_{n-2}X_{n-1}X_n = HTHH\}$.

Problem 3. Given a filtration $\{\mathcal{F}_n\}$, i.e., $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}$, we define $\mathcal{F}_\infty = \sigma(\cup_{n=1}^\infty \mathcal{F}_n)$.

(1) Is it correct that $\mathcal{F}_\infty = \cup_{n=1}^\infty \mathcal{F}_n$? If not, please give a counterexample.

(2) Let X be a random variable which is $\mathcal{F}$-measurable and integrable. Prove $\{\mathbb{E}(X|\mathcal{F}_n)\}_{n\geq 1}$ is uniformly integrable.

(3) Prove $\mathbb{E}(X|\mathcal{F}_n) \to \mathbb{E}(X|\mathcal{F}_\infty)$ in L^1, as n goes to infinity.

Problem 4. Consider the least squares problem. Assume Y is the n-dimensional outcome vector and X is the $n \times p$ covariate/design matrix. Assume X is full rank. We can run least squares of Y on X to obtain the usual estimator $\hat{\beta}$, the residual vector $\hat{\varepsilon}$, and the hat matrix $H = X(X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}$.

Now we want to compute the least square coefficient $\hat{\beta}_{[i]}$ by dropping the ith observation, $i = 1, \ldots, n$. Instead of running the least squares n times, can we obtain $\{\hat{\beta}_{[i]},\ i = 1, \ldots, n\}$ from $\hat{\beta}$, $\hat{\varepsilon}$, $(X^{\mathrm{T}}X)^{-1}$, and H, so that we only need to run the least squares only once?